PMM U.S.S.R.,Vol.45,pp.464-467
0021-8928/82/4 0464 $7.50/0

UDC 532.516

STABILITY OF FLOW OF A VISCOUS INCOMPRESSIBLE FLUID ALONG AN ELASTIC WALL*

M.P. MIASNIKOV

Stability relative to small long-wave disturbances of the flow of a heavy viscous incompressible fluid interacting with a wall coated with a layer of incompressible elastic material is considered.

1. Two stability problems for long-wave type disturbances are studied: (1) plane-parallel flow of fluid draining along an inclined plane coated with a layer of elastic material; (2) flow of fluid draining along the outer surface of a vertical tube with circular cross-section coated with a layer of elastic material.

The motion of a viscous incompressible fluid is described by the Navier–Stokes equations /1/

$$\frac{\partial v^k}{\partial t} + v^j\nabla_j v^k = g^k - \frac{1}{\rho_1} g^{kj}\frac{\partial p_1}{\partial x_j} + \nu\Delta v^k, \quad \nabla_k v^k = 0$$

where v^k are the contravariant components of the velocity vector; ρ_1 density of fluid; p_1 pressure in fluid; ν kinematic viscosity of fluid; and g^k contravariant components of the free fall acceleration vector. The system of equations of the motion of an elastic material has the form /2,3/

$$\frac{\partial u^k}{\partial t} + u^j\nabla_j u^k = g^k + \frac{1}{\rho_2}\nabla_j p^{kj}, \quad \nabla_k u^k = 0$$
$$\frac{\partial \varepsilon_{ij}}{\partial t} + u^k\nabla_k\varepsilon_{ij} + \varepsilon_{ki}\nabla_j u^k + \varepsilon_{kj}\nabla_i u^k = e_{ij}$$
$$e_{ij} = {}^1/_2(g_{kj}\nabla_i u^k + g_{ki}\nabla_j u^k), \quad p^{kj} = -p_2 g^{kj} + 2\mu g^{k\alpha}g^{j\beta}\varepsilon_{\alpha\beta}$$

where u^k are the contravariant components of the velocity vector of points in the elastic material; p^{kj} contravariant components of the stress tensor; $\varepsilon_{\alpha\beta}$ contravariant components of the deformation tensor; ρ_2 density of elastic material; p_2 pressure in elastic material; and μ shear modulus.

The boundary conditions are as follows:

(1) on the free surface of the fluid,

$$F(x_1, x_2, x_3, t) = 0, \quad \frac{dF}{dt} = \frac{\partial F}{\partial t} + v^i\frac{\partial F}{\partial x_i} = 0$$
$$(-p_1 g^{kj} + 2\rho_1 g^{k\alpha}g^{j\beta}e_{\alpha\beta})\frac{\partial F}{\partial x_j} = -p\frac{\partial F}{\partial x_k}$$

where $-p$ is atmospheric pressure $(p > 0)$;

(2) on the interface between the fluid and the elastic material,

$$f(x_1, x_2, x_3, t) = 0, \quad (-p_1 g^{kj} + 2\rho_1\nu g^{k\alpha}g^{j\beta}e_{\alpha\beta})\frac{\partial f}{\partial x_j} = (-p_2 g^{kj} + 2\mu g^{k\alpha}g^{j\beta}\varepsilon_{\alpha\beta})\frac{\partial f}{\partial x_j}, \quad u^i = v^i$$

(3) fixed attachment condition between the elastic material and the solid wall,

$$\varphi(x_1, x_2, x_3) = 0, \quad u^i = 0$$

All the equations and boundary conditions are described below for physical components of the vectors and tensors in a dimensionless form. The physical components of the velocity vector v_i and u_i in an arbitrary orthogonal curvilinear system are expressed in terms of the contravariant components v^i and u^i as follows /1/: $v_i = v^i\sqrt{g_{ii}}$, $u_i = u^i\sqrt{g_{ii}}$, where g_{ij} are the contravariant components of the metric tensor. We have /1/ $\sigma_{ij} = p^{ij}\sqrt{g_{ii}}\sqrt{g_{jj}}$, $s_{\alpha\beta} = \varepsilon_{\alpha\beta}\sqrt{g^{\alpha\alpha}}\sqrt{g^{\beta\beta}}$ for the physical components of the stress and deformation tensor; here, g^{ij} are the contravariant components of the metric tensor. In passing to dimensionless variables, the characteristic

*Prikl.Matem.Mekhan.,45,No.4,631-636,1981 length l_0, velocity V_0 and free fall acceleration g serve as the scale. The stress tensor components are related to the quantity $\rho_1 V_0^2$, and time is measured in the scale $t_0 = l_0/V_0$. The following dimensionless parameters occur in the equations: $R = V_0 l_0/\nu$ the Reynolds number; $F = V_0^2/gl_0$ the Froude number; and $m = V_0 \sqrt{\rho_2/\mu}$, $\varkappa = \rho_1/\rho_2$.

2. The drainage problem for a heavy viscous incompressible fluid flowing on an inclined plane coated with a layer of elastic material has the stationary solution

$$\begin{aligned}
&v_x^\circ(y) = -{}^1/_2\, q\,(y-h_0)^2 + q\,(H_0-h_0)\,(y-h_0),\ v_y^\circ = 0\\
&p_1^\circ(y) = p + r\,(H_0 - y),\ u_x^\circ = u_y^\circ = s_{xx}^\circ = 0\\
&s_{xy}^\circ(y) = G\,[h_0 - y + \varkappa\,(H_0-h_0)]\sin\theta,\ \ s_{yy}^\circ = -2[s_{xy}^\circ]^2\\
&\varkappa m^2 p_2^\circ(y) = \varkappa m^2 p + 2G[h_0 - y + \varkappa\,(H_0-h_0)]\cos\theta -\\
&\quad 4G^2\,[h_0 - y + \varkappa\,(H_0-h_0)]^2\sin^2\theta\\
&G = {}^1/_2 m^2 F^{-1},\quad q = RF^{-1}\sin\theta,\ r = F^{-1}\cos\theta
\end{aligned}$$

Here v_x°, v_y° are the components of the velocity vector in the fluid; u_x°, u_y° are the components of the velocity vector in the elastic material; p_1° and p_2° pressure in the fluid and in the elastic material, respectively; and $s_{xx}^\circ, s_{yy}^\circ, s_{xy}^\circ$ are the components of the deformation tensor in the elastic material. The x-axis of a rectangular Cartesian coordinate system has the same direction as the motion of the fluid, while the y-axis is directed towards the free surface of the fluid $y = H_0$. The interface between the fluid and the elastic material $y = h_0$, and the angle of inclination of the plane to the horizon is θ.

To achieve compressed notation, we introduce a correspondence between the numerical and literal indices:

$$\left\{\begin{matrix} x,\ y \to i & x & y\\ & 1 & 2\end{matrix}\right\}$$

To study the stability of steady-state motion relative to small disturbances, we set /4/

$$\begin{aligned}
&v_i = v_i^\circ + \alpha v_i^1,\quad u_i = u_i^\circ + \alpha u_i^1,\quad s_{ij} = s_{ij}^\circ + \alpha s_{ij}^1\\
&p_1 = p_1^\circ + \alpha p_1^1,\quad p_2 = p_2^\circ + \alpha p_2^1
\end{aligned}$$

where $\alpha \ll 1$ and represent the disturbances in the form /4/

$$\{v_i^1, u_i^1, p_1^1, p_2^1, s_{ij}^1\} = \{\varphi_i(y), \Phi_i(y), \varphi_3(y), \Phi_3(y), \Psi_{ij}(y)\}\exp[i\omega(x - ct)]$$

In the case of long-wave type disturbances, we have $\omega \ll 1$. We obtain the following boundary-value problem for the disturbances (the prime denotes the derivative with respect to y):

$$\begin{aligned}
&\varphi_2'''' - i\omega R\,(v-c)\,\varphi_2'' + i\omega R v''\varphi_2 = 0\\
&\Phi_2'''' + 4i\omega s\Phi_2''' + 6i\omega s'\Phi_2'' = 0\\
&y = H_0\colon\ (v_m - c)\,\varphi_2''' - i\omega R\,(v_m - c)^2\varphi_2' + i\omega R r\varphi_2 = 0\\
&(v_m - c)\varphi_2'' - v''\varphi_2 = 0\\
&y = h_0\colon\ i\omega c\varkappa m^2\varphi_2'' + i\omega\varkappa m^2 v''\varphi_2 = -R\Phi_2''\\
&i\omega c\varkappa m^2\varphi_2''' = -R\Phi_2''' - 4i\omega Rs\Phi_2'' - 2i\omega Rs'\Phi_2'\\
&c\varphi_2' + v'\varphi_2 = c\Phi_2',\ \varphi_2 = \Phi_2\\
&y = 0\colon\ \Phi_2 = 0,\ \Phi_2' = 0\\
&v = v_x^\circ(y),\quad v_m = v\,(H_0),\ s = s_{xy}^\circ(y)
\end{aligned}$$

Terms on the order of ω^2 have been omitted.

We will find solutions of the equations for the function $\varphi_2(y)$ and $\Phi_2(y)$ in the form of expansion in series in powers of ω, obtaining

$$\begin{aligned}
\varphi_2(y) = {}&c_1 + c_2(y-h_0) + c_3\Big\{(y-h_0)^2 + i\omega R\Big[-c\frac{(y-h_0)^4}{12} + q\,(H_0-h_0)\frac{(y-h_0)^5}{60}\Big]\Big\} + c_4\Big\{(y-h_0)^3 +\\
&i\omega R\Big[-c\frac{(y-h_0)^5}{20} + q\,(H_0-h_0)\frac{(y-h_0)^6}{60} - q\frac{(y-h_0)^7}{420}\Big]\Big\}
\end{aligned}$$

$$\Phi_2(y) = b_1 + b_2 y + b_3 y^2 + b_4\{y^3 - i\omega G[h_0 + \varkappa\,(H_0-h_0)]\,y^4\sin\theta + {}^1/_5\, i\omega G y^5\sin\theta\}$$

The boundary conditions yield eight linear homogeneous equations for the constants c_n, b_n $(n = 1, 2, 3, 4)$. Since the determinant of the system is equal to zero, we are led to the following equation for c:

$$\begin{aligned}
&6\,(2v_m - c) - 2i\omega Rr\,(H_0-h_0)^3 + 3i\omega R\Big[(v_m - c)^2(H_0-h_0)^2 -\\
&v_m(v_m - c)\frac{(H_0-h_0)^2}{6} - v_m^2\frac{(H_0-h_0)^2}{10}\Big] -\\
&6i\omega\varkappa m^2 q h_0 R^{-1}(H_0-h_0)\,c = 0
\end{aligned}$$

In order to satisfy this equation we require $c = c^\circ + \omega c^1$, whence we find

$$c^\circ = 2v_m, \quad c^1 = -\frac{1}{3} iRr (H_0 - h_0)^3 + \frac{8}{15} iR (H_0 - h_0)^2 v_m^2 - 2i\varkappa m^2 q h_0 R^{-1} (H_0 - h_0) v_m$$

Suppose the equation of the free surface of the liquid in undisturbed motion has the form $y = H$ in dimensional magnitudes, while the interface between the fluid and the elastic coating is at $y = h$. We set $l_0 = H - h$, $V_0 = \sqrt{g(H-h)}$, whence $F = 1$, $R = (H-h)\sqrt{g(H-h)}/\nu$, $m^2 = \rho_2 g (H-h)/\mu$. The stability condition imposed on the mainstream flow will have the form

$$R^2 < \frac{5}{2} \frac{\cos\theta}{\sin^2\theta} + \frac{15}{2} \frac{\rho_1 g h}{\mu}$$

The elastic coating stabilizes the mainstream flow. In the case of a vertical plane $(\theta = \pi/2)$ the elastic coating creates a fluid flow stability region, whereas flow is unstable in the case of a solid wall. The more elastic the material of the coating the greater the stability region. If $h = 0$ or $\mu \to \infty$ (no elastic coating), we obtain the stability condition given in /5/.

3. The drainage problem for a heavy viscous incompressible fluid flowing on the outer surface of a vertical tube coated with a layer of elastic material has the steady-state solution

$$\begin{aligned}
&v_z^\circ(r) = {}^1/_4\, q(r^2 - h_0^2) - {}^1/_2 q H_0^2 (\ln r - \ln h_0)\\
&p_1^\circ = p, \quad v_r^\circ = v_\varphi^\circ = u_r^\circ = u_\varphi^\circ = u_z^\circ = s_{zz}^\circ = 0\\
&s_{rz}^\circ(r) = {}^1/_2 G (r - \lambda r^{-1}), \quad s_{rr}^\circ(r) = -{}^1/_2\, G^2 (r - \lambda r^{-1})^2\\
&\varkappa m^2 p_2^\circ(r) = \varkappa m^2 p + {}^1/_4 G^2 [-6r^2 + 2h_0^2 +\\
&\quad 8\lambda (1 + \ln r - \ln h_0) - 2\lambda^2 (r^{-2} + h_0^{-2})]\\
&q = RF^{-1}, \lambda = h_0^2 + \varkappa (H_0^2 - h_0^2), \quad G = {}^1/_2\, m^2 F^{-1}
\end{aligned}$$

Here r, φ, and z constitutes a cylindrical coordinate system whose z axis coincides with the longitudinal axis of the tube; r, φ are polar coordinates in the cross-sectional plane; the outer surface of the solid wall of the tube is $r = a_0$, the interface between the elastic coating and the fluid $r = h_0$, and the free surface of the fluid $r = H_0$. The rest of the notation has the same meaning as above. To abbreviate the notation, we introduce a correspondence between the literal and numerical indices:

$$\left\{ \begin{matrix} r, \varphi, z \to i & r \ \varphi \ z \\ & 1 \ 2 \ 3 \end{matrix} \right\}$$

To study the stability of steady-state motion relative to small disturbances, we set /4/

$$\begin{aligned}
&v_i = v_i^\circ + \alpha v_i^1, \quad u_i = u_i^\circ + \alpha u_i^1, \quad s_{ij} = s_{ij}^\circ + \alpha s_{ij}^1\\
&p_1 = p_1^\circ + \alpha p_1^1, \ p_2 = p_2^\circ + \alpha p_2^1
\end{aligned}$$

where $\alpha \ll 1$, and represent the disturbances in the form /4/

$$\{v_i^1, u_i^1, p_1^1, p_2^1, s_{ij}^1\} = \{\varphi_i(r), \Phi_i(r), \varphi_4(r), \Phi_4(r), \Psi_{ij}(r)\} \exp [i\omega (z - ct)]$$

Setting $\omega \ll 1$, we obtain the boundary-value problem (the prime denotes the derivative with respect to r)

$$\varphi_1'''' + 2r^{-1}\varphi_1''' - 3r^{-2}\varphi_1'' + 3r^{-3}\varphi_1' - 3r^{-4}\varphi_1 - i\omega R(v - c)(\varphi_1'' + r^{-1}\varphi_1' - r^{-2}\varphi_1) + i\omega R v''\varphi_1 - i\omega R r^{-1} v'\varphi_1 = 0$$

$$\begin{aligned}
&\Phi_1'''' + (2r^{-1} + 4i\omega s)\Phi_1''' + (-3r^{-2} + 6i\omega r^{-1}s + 6i\omega s')\Phi_1'' + (3r^{-3} + 4i\omega s'' - 2i\omega r^{-2}s + 2i\omega r^{-1}s')\Phi_1' + (-3r^{-4} +\\
&\quad 2i\omega s''' - 4i\omega r^{-2}s' + 4i\omega r^{-3}s)\Phi_1 = 0
\end{aligned}$$

$$r = H_0\colon \ \varphi_1''' + 2H_0^{-1}\varphi_1'' - H_0^{-2}\varphi_1' + H_0^{-3}\varphi_1 - i\omega R(v_m - c)\varphi_1' - i\omega R H_0^{-1}(v_m - c)\varphi_1 = 0$$

$$\varphi_1'' + H_0^{-1}\varphi_1' - [H_0^{-2} + v''(v_m - c)^{-1}]\varphi_1 = 0$$

$$\begin{aligned}
&r = h_0\colon \ i\omega\varkappa m^2 c R^{-1}(\varphi_1''' + 2h_0^{-1}\varphi_1'' - h_0^{-2}\varphi_1' + h_0^{-3}\varphi_1) =\\
&\quad -\Phi_1''' - (2h_0^{-1} + 4i\omega s)\Phi_1'' + (h_0^{-2} - 2i\omega s' - 2i\omega h_0^{-1}s)\Phi_1' + (-h_0^{-3} - 2i\omega s'' + 4i\omega h_0^{-2}s)\Phi_1
\end{aligned}$$

$$i\omega\varkappa m^2 R^{-1}[c(\varphi_1'' + h_0^{-1}\varphi_1' - h_0^{-2}\varphi_1) + v''\varphi_1] = -\Phi_1'' - h_0^{-1}\Phi_1' + h_0^{-2}\Phi_1 + 2i\omega h_0^{-1}s\Phi_1$$

$$c\varphi_1' + v'\varphi_1 = c\Phi_1', \ \varphi_1 = \Phi_1, \quad r = a_0\colon \Phi_1 = 0, \ \Phi_1' = 0, \quad v = v_z^\circ(r), \ v_m = v(H_0), \ s = s_{rz}^\circ(r)$$

We find the solutions of the equations for $\varphi_1(r)$ and $\Phi_1(r)$ in the form of a series in powers of ω, we obtain

$$\varphi_1(r)=c_1r+c_2\left\{r^3+\frac{i\omega R}{144}\left[6r^5(v-c)+\frac{11}{4}qH_0^2r^5-\frac{5}{4}qr^7\right]\right\}+c_3\left[r^{-1}+\frac{1}{8}i\omega RqH_0^2r(\ln r-\ln h_0)^2\right]+c_4\left\{r(\ln r-\ln h_0)+\frac{i\omega R}{96}\left[\frac{1}{4}qr^5+(12r^3v-12r^3c+15qH_0^2r^3-3qr^5)(\ln r-\ln h_0)\right]\right\}$$

$$\Phi_1(r)=b_1r+b_2[r^3-{}^1/_4\, i\omega Gr^5+i\omega G\lambda r^3(\ln r-\ln h_0)]+b_3[r^{-1}+i\omega G\lambda r^{-1}(\ln r-\ln h_0)]+b_4r(\ln r-\ln h_0)$$

Proceeding entirely analogously to the analysis in Sect.2, we obtain the stability condition in the form

$$R^2h_0^2f(\beta)<192\,\varkappa m^2g(\beta)$$

$$f(\beta)=128\beta^6-36\beta^4-144\beta^2+52+(96\beta^6-720\beta^4+288\beta^2)\ln\beta-480\beta^4(\beta^2-1)\ln^2\beta+384\beta^6\ln^3\beta$$

$$g(\beta)=(\beta^4-1)(1-\gamma^2)+(6\beta^4-8\beta^2+2)\ln\gamma-4(1-\gamma^2)\beta^2\ln\beta-8\beta^4\ln\beta\ln\gamma$$

$$\beta=H_0/h_0>1,\quad \gamma=a_0/h_0<1$$

The resulting stability condition is valid if the radius of curvature of the cylindrical surface of the tube is of the order of the characteristic linear dimension of the problem.

If it is assumed that the radius of curvature of the tube cylindrical surface is of the order of magnitude of the wavelength or greater, the order of the terms occuring in the equations and the boundary conditions of the problem change, which leads to a rearrangement of the asymptotic series for the functions $\varphi_1(r)$ and $\Phi_1(r)$. A passage to the limit with $a_0\to\infty$ in the resulting formula, therefore, is without meaning and yields the incorrect answer $R=0$. The case in which the radius of curvature of the surface of the tube is of the order of magnitude of the wavelength may be studied using the method of Sect.2. If $a_0=0$, the fluid drains along the surface of a solid elastic cylinder. To study the stability of such flow, it is necessary to take into account longitudinal stretching of the elastic cylinder.

Suppose that the equation of the free surface of the fluid in undisturbed motion has the form $r=H$ in dimensional variables, the equation of the interface between the elastic coating and the fluid has the form $r=h$, and the equation of the tube surface, $r=a$. We select $l_0=h,\ V_0=\sqrt{gh}$. Then, $\beta=H/h,\ \gamma=a/h,\ R=h\sqrt{gh}/\nu$. It can be shown that

$$f(1)=f'(1)=f''(1)=f'''(1)=0,\quad f''''(1)=2304$$
$$g(1)=g'(1)=g''(1)=0,\quad g'''(1)=16(1-\gamma^2-4\ln\gamma)>0$$

Consequently, if $H-h\ll h\ (\beta\to1)$, we will have /6/

$$f(\beta)=\frac{f''''(1)+\alpha_1(\beta)}{4!}(\beta-1)^4,\quad g(\beta)=\frac{g'''(1)+\alpha_2(\beta)}{3!}(\beta-1)^3$$

where $\alpha_1(\beta)\to0$ and $\alpha_2(\beta)\to0$ as $\beta\to1$. The stability condition, assuming $H-h\ll h$, will have the form

$$R^2<\frac{16}{3}\frac{\rho_1g}{\mu}[1-4\ln\gamma-\gamma^2]\frac{h^2}{H-h}$$

The resulting formula shows that flow is always unstable if $h=a$ or as $\mu\to\infty$ (no elastic coating). The elastic coating creates a stability reserve for the mainstream flow, and the smaller the quantity μ, i.e., the more elastic the material of the coating, the greater the stability reserve.

REFERENCES

1. SEDOV L.I., Mechanics of Continuous Medium. Vol.1. Moscow, NAUKA, 1976. See also, in English, A Course in Mechanics of Continuum, Wolters-Noordhoff, 1971.
2. SEDOV L.I., Introduction to Continuum Mechanics. Moscow, FIZMATGIZ, 1962.
3. GODUNOV S.K., Elements of Continuum Mechanics. Moscow, NAUKA, 1978.
4. LIN CHIA-CHIAO., The Theory of Hydrodynamic Stability. Cambridge Univ. Press, 1955.
5. IVANILOV Iu.P., On the stability of plane parallel flow of a viscous fluid over an inclined bottom. PMM. Vol.24, No.2, 1960.
6. FIKHTENGOL'TS G.M., Course in Differential and Integral Calculus. Vol.1, Moscow, NAUKA, 1969.

Translated by R.H.S.